

Cvičení č. 13 z KMA-MMAN2

19. a 22. února 2009

1 Úlohy na přímou integraci

Úloha 1.1. $I = \int \frac{5x^2 - 3}{\sqrt{x}} dx$

Řešení. $I = \int \frac{5x^2 - 3}{\sqrt{x}} dx = \int \left(\frac{5x^2}{\sqrt{x}} - \frac{3}{\sqrt{x}} \right) dx =$

$$5 \int x^{2-\frac{1}{2}} dx - 3 \int x^{-\frac{1}{2}} dx = \frac{5}{\frac{5}{2}} x^{\frac{5}{2}} + \frac{3}{\frac{1}{2}} x^{\frac{1}{2}} + C = 2x^{\frac{5}{2}} + 6x^{\frac{1}{2}} + C. \quad \square$$

Úloha 1.2. $I = \int \frac{3^x \cos^2 x - 5}{\cos^2 x} dx$

Řešení. $I = \int \frac{3^x \cos^2 x - 5}{\cos^2 x} dx = \int 3^x dx - 5 \int \frac{dx}{\cos^2 x} dx = \dots . \quad \square$

2 Úlohy na jednoduché substituce

2.1 Lineární substituce $u = ax + b$

Úloha 2.1. $I = \int (\cos(2x - 3) + \sin(x + 5)) dx.$

Řešení. $I = \int (\cos(2x - 3) + \sin(x + 5)) dx = \int \cos(2x - 3) dx + \int \sin(x + 5) dx =$

$$\left[\begin{array}{l|l} u = 2x - 3 & v = x + 5 \\ du = 2 dx & dv = dx \\ \frac{du}{2} = dx & \end{array} \right] = \frac{1}{2} \int \cos(u) du + \int \sin(v) dv =$$

$$\frac{1}{2} \sin(u) - \cos(v) + C = \frac{1}{2} \sin(2x - 3) - \cos(x + 5) + C. \quad \square$$

2.1.1 Další jednoduché substituce

Úloha 2.2. $I = \int \frac{dx}{e^x - 1}$.

Řešení. $I = \int \frac{dx}{e^x - 1} = \left[\begin{array}{l} u = e^x \\ du = e^x dx \\ du = u dx \\ \frac{du}{u} = dx \end{array} \right] = \int \frac{\frac{du}{u}}{u - 1} = \int \frac{1}{u(u - 1)} du =$
 $-\int \frac{-1}{u(u - 1)} du = -\int \frac{-u + u - 1}{u(u - 1)} du = -\int \frac{(-u) + (u - 1)}{u(u - 1)} du =$
 $-\int \left(-\frac{u}{u(u - 1)} + \frac{u - 1}{u(u - 1)} \right) du = -\int \left(-\frac{1}{u - 1} + \frac{1}{u} \right) du =$
 $\int \frac{1}{u - 1} du - \int \frac{1}{u} du = \ln |u - 1| - \ln |u| + C =$
 $\ln |e^x - 1| - \ln |e^x| + C = \ln |e^x - 1| - \ln e^x + C = \ln |e^x - 1| - x + C. \quad \square$

Úloha 2.3. $I = \int \frac{x dx}{\sqrt{x^2 - a^2}}$.

Řešení. $I = \int \frac{x dx}{\sqrt{x^2 - a^2}} = \left[\begin{array}{l} u = x^2 - a^2 \\ du = 2x dx \\ \frac{du}{2} = x dx \end{array} \right] = \int \frac{\frac{du}{2}}{\sqrt{u}} = \frac{1}{2} \int u^{-\frac{1}{2}} du =$
 $\frac{1}{2} \frac{1}{-\frac{1}{2}} u^{\frac{1}{2}} + C = \sqrt{u} + C = \sqrt{x^2 - a^2} + C. \quad \square$

3 Integrace metodou per partes

$$(u \cdot v)' = u'v + uv' \implies uv' = (u \cdot v)' - u'v \implies$$
$$\int uv' dx = \int (u \cdot v)' dx - \int u'v dx = uv - \int u'v dx.$$

Úloha 3.1. $I = \int \sin^2 x dx$.

Řešení. $I = \int \sin^2 x dx = \left[\begin{array}{ll} u = \sin x & v' = \sin x \\ u' = \cos x & v = -\cos x \end{array} \right] = -\sin x \cos x + \int \cos^2 x dx =$
 $-\sin x \cos x + \int (1 - \sin^2 x) dx = -\sin x \cos x + \int dx - \int \sin^2 x dx = -\sin x \cos x +$
 $x - \int \sin^2 x dx.$

Máme tedy rovnost:

$$\int \sin^2 x \, dx = -\sin x \cos x + x - \int \sin^2 x \, dx.$$

Odtud:

$$\int \sin^2 x \, dx = \frac{x - \sin x \cos x}{2} + C.$$

□

Úloha 3.2. $I = \int x^2 \cos 2x \, dx.$

$$\text{Řešení. } \int x^2 \cos 2x \, dx = \left[\begin{array}{ll} u = x^2 & v' = \cos 2x \\ u' = 2x & v = \frac{1}{2} \sin 2x \end{array} \right] =$$

$$x^2 \frac{1}{2} \sin 2x - \int 2x \frac{1}{2} \sin 2x \, dx = \frac{x^2}{2} \sin 2x - \int x \sin 2x \, dx =$$

$$\left[\begin{array}{ll} u = x & v' = \sin 2x \\ u' = 1 & v = -\frac{1}{2} \cos 2x \end{array} \right] = \frac{x^2}{2} \sin 2x + x \frac{1}{2} \cos 2x - \frac{1}{2} \int \cos 2x \, dx =$$

$$\frac{x^2}{2} \sin 2x + \frac{x}{2} \cos 2x - \frac{1}{2} \frac{1}{2} \sin 2x + C = \frac{x^2}{2} \sin 2x + \frac{x}{2} \cos 2x - \frac{1}{4} \sin 2x + C. \quad \square$$

Úloha 3.3. $I = \int \operatorname{tg}^2 x \, dx.$

$$\text{Řešení. } I = \int \operatorname{tg}^2 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} \, dx =$$

$$\left[\begin{array}{ll} u = \sin^2 x & v' = \frac{1}{\cos^2 x} \\ u' = 2 \sin x \cos x & v = \operatorname{tg} x = \frac{\sin x}{\cos x} \end{array} \right] = \sin^2 x \frac{\sin x}{\cos x} - \int 2 \sin x \cos x \frac{\sin x}{\cos x} \, dx =$$

$$\frac{\sin^3 x}{\cos x} - 2 \int \sin^2 x \, dx = \left[\int \sin^2 x \, dx = \frac{x - \sin x \cos x}{2} + C \right] =$$

(Úloha 3.1)

$$\frac{\sin^3 x}{\cos x} - 2 \frac{x - \sin x \cos x}{2} + C = \frac{\sin^3 x}{\cos x} - x + \sin x \cos x + C. \quad \square$$

3.1 Výpočet I_c a I_s

Úloha 3.4. *Vypočtěte*

$$I_c = \int e^{ax} \cos bx \, dx, \quad I_s = \int e^{ax} \sin bx \, dx$$

(budeme počítat primitivní funkce pro $C = 0$).

Řešení. V integrálu I_c se použije dvěma způsoby metoda per partes: a) pro $u = \cos bx$ a $v' = e^{ax}$, b) pro $u = e^{ax}$, $v' = \cos bx$:

$$\text{a) } I_c = \int e^{ax} \cos bx \, dx = \left[\begin{array}{ll} u = \cos bx & v' = e^{ax} \\ u' = -b \sin bx & v = \frac{1}{a} e^{ax} \end{array} \right] = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \, dx = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} I_s.$$

$$\text{b) } I_c = \int e^{ax} \cos bx \, dx = \left[\begin{array}{ll} u = e^{ax} & v' = \cos bx \\ u' = a e^{ax} & v = \frac{1}{b} \sin bx \end{array} \right] = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} I_s.$$

Tím dostaneme soustavu

$$I_c = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} I_s, \quad I_c = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} I_s,$$

tedy

$$\begin{aligned} \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} I_s &= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} I_s, \\ \frac{b}{a} I_s + \frac{a}{b} I_s &= \frac{1}{b} e^{ax} \sin bx - \frac{1}{a} e^{ax} \cos bx \quad | \cdot ab, \\ (b^2 + a^2) I_s &= a e^{ax} \sin bx - b e^{ax} \cos bx, \end{aligned}$$

odtud

$$\begin{aligned} I_s &= \frac{a \sin bx - b \cos bx}{a^2 + b^2} e^{ax}. \\ I_c &= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} I_s = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \frac{a \sin bx - b \cos bx}{a^2 + b^2} e^{ax} \\ &= \frac{(a^2 + b^2) \cos bx + ab \sin bx - b^2 \cos bx}{a(a^2 + b^2)} e^{ax} = \frac{b \sin bx + a \cos bx}{a^2 + b^2} e^{ax} \end{aligned}$$

□

3.2 Rekurentní vzorec pro integrál I_n

Úloha 3.5. Nalezněte rekurentní vzorec pro integrál $I_n = \int \frac{dx}{(a^2 + x^2)^n}$, $n \geq 2$.

Řešení. V integrálu I_m , kde $m \geq 1$, položíme $u = \frac{1}{(a^2 + x^2)^n}$, $v' = 1$ a dostaneme

$I_m = \frac{x}{(a^2 + x^2)^m} + 2m I_m - 2ma^2 I_{m+1}$, odkud vyjádříme I_{m+1} . Položíme-li pak $m = n - 1$, dostaneme

$$I_n = \frac{1}{2(n-1)a^2} \frac{x}{(a^2 + x^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1}.$$

□

Úloha 3.6. Vypočtěte $I = \int \frac{dx}{(2^2 + x^2)^3}$.

Řešení. Využijeme rekurentní vzorec pro I_3 :

$$\begin{aligned}
 I &= I_3 = \frac{1}{2(3-1)2^2} \frac{x}{(2^2 + x^2)^{3-1}} + \frac{2 \cdot 3 - 3}{2(3-1)2^2} I_{3-1} \\
 &= \frac{1}{16} \frac{x}{(2^2 + x^2)^2} + \frac{3}{16} \int \frac{dx}{(2^2 + x^2)^2} \\
 &= \frac{1}{16} \frac{x}{(2^2 + x^2)^2} + \frac{3}{16} \left(\frac{1}{2(2-1)2^2} \frac{x}{(2^2 + x^2)^{2-1}} + \frac{2 \cdot 2 - 3}{2(2-1)2^2} I_{2-1} \right) \\
 &= \frac{1}{16} \frac{x}{(2^2 + x^2)^2} + \frac{3}{16} \left(\frac{1}{8} \frac{x}{2^2 + x^2} + \frac{1}{8} \int \frac{dx}{2^2 + x^2} \right) \\
 &= \frac{1}{16} \frac{x}{(2^2 + x^2)^2} + \frac{3}{16} \left(\frac{1}{8} \frac{x}{2^2 + x^2} + \frac{1}{8} \left(\frac{1}{2^2} \int \frac{dx}{1 + \left(\frac{x}{2}\right)^2} \right) \right) \\
 &= \frac{1}{16} \frac{x}{(2^2 + x^2)^2} + \frac{3}{16} \left(\frac{1}{8} \frac{x}{2^2 + x^2} + \frac{1}{8} \left(\frac{1}{2^2} \frac{1}{\frac{1}{2}} \operatorname{arctg} \frac{x}{2} \right) \right) + C \\
 &= \frac{1}{16} \left(\frac{x}{(2^2 + x^2)^2} + \frac{3}{8} \frac{x}{2^2 + x^2} + \frac{3}{16} \operatorname{arctg} \frac{x}{2} \right) + C.
 \end{aligned}$$

□