Cvičení č. 13 z KMA-MMAN2

15. a 18. února 2010

1 Úlohy na přímou integraci

Úloha 1.1. $I = \displaystyle\int \frac{5x^2 - 3}{\sqrt{x}}\,\mathrm{d}x$

Řešení. $I = \displaystyle\int \frac{5x^2 - 3}{\sqrt{x}}\,\mathrm{d}x = \int \left(\frac{5x^2}{\sqrt{x}} - \frac{3}{\sqrt{x}}\right)\mathrm{d}x =$

$$5\int x^{2-\frac{1}{2}}\,\mathrm{d}x - 3\int x^{-\frac{1}{2}}\,\mathrm{d}x = \frac{5}{\frac{5}{2}}x^{\frac{5}{2}} + \frac{3}{\frac{1}{2}}x^{\frac{1}{2}} + C = 2x^{\frac{5}{2}} + 6x^{\frac{1}{2}} + C. \qquad \square$$

Úloha 1.2. $I = \displaystyle\int \frac{3^x \cos^2 x - 5}{\cos^2 x}\,\mathrm{d}x$

Řešení. $I = \displaystyle\int \frac{3^x \cos^2 x - 5}{\cos^2 x}\,\mathrm{d}x = \int 3^x\,\mathrm{d}x - 5\int \frac{\mathrm{d}x}{\cos^2 x}\,\mathrm{d}x = \cdots .$ $\square$

2 Úlohy na jednoduché substituce

2.1 Lineární substituce $u = ax + b$

Úloha 2.1. $I = \displaystyle\int \Big(\cos(2x-3) + \sin(x+5)\Big)\,\mathrm{d}x.$

Řešení. $I = \displaystyle\int \Big(\cos(2x-3)+\sin(x+5)\Big)\,\mathrm{d}x = \int \cos(2x-3)\,\mathrm{d}x + \int \sin(x+5)\,\mathrm{d}x =$

$$\left[\begin{array}{c|c} u = 2x-3 & v = x+5 \\ \mathrm{d}u = 2\,\mathrm{d}x & \mathrm{d}v = \mathrm{d}x \\ \frac{\mathrm{d}u}{2} = \mathrm{d}x & \end{array}\right] = \frac{1}{2}\int \cos(u)\mathrm{d}u + \int \sin(v)\mathrm{d}v =$$

$$\frac{1}{2}\sin(u) - \cos(v) + C = \frac{1}{2}\sin(2x-3) - \cos(x+5) + C. \qquad \square$$

2.1.1 Další jednoduché substituce

Úloha 2.2. $I = \displaystyle\int \frac{\mathrm{d}x}{\mathrm{e}^x - 1}$.

Řešení. $I = \displaystyle\int \frac{\mathrm{d}x}{\mathrm{e}^x - 1} = \begin{bmatrix} u = \mathrm{e}^x \\ \mathrm{d}u = \mathrm{e}^x\,\mathrm{d}x \\ \mathrm{d}u = u\,\mathrm{d}x \\ \frac{\mathrm{d}u}{u} = \mathrm{d}x \end{bmatrix} = \int \frac{\frac{\mathrm{d}u}{u}}{u-1} = \int \frac{1}{u(u-1)}\mathrm{d}u =$

$-\displaystyle\int \frac{-1}{u(u-1)}\mathrm{d}u = -\int \frac{-u+u-1}{u(u-1)}\mathrm{d}u = -\int \frac{(-u)+(u-1)}{u(u-1)}\mathrm{d}u =$

$-\displaystyle\int \Big(-\frac{u}{u(u-1)} + \frac{u-1}{u(u-1)}\Big)\mathrm{d}u = -\int \Big(-\frac{1}{u-1} + \frac{1}{u}\Big)\mathrm{d}u =$

$\displaystyle\int \frac{1}{u-1}\mathrm{d}u - \int \frac{1}{u}\mathrm{d}u = \ln|u-1| - \ln|u| + C =$

$\ln|\mathrm{e}^x - 1| - \ln|\mathrm{e}^x| + C = \ln|\mathrm{e}^x - 1| - \ln \mathrm{e}^x + C = \ln|\mathrm{e}^x - 1| - x + C.$ □

Úloha 2.3. $I = \displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt{x^2 - a^2}}$.

Řešení. $I = \displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt{x^2-a^2}} = \begin{bmatrix} u = x^2 - a^2 \\ \mathrm{d}u = 2x\,\mathrm{d}x \\ \frac{\mathrm{d}u}{2} = x\,\mathrm{d}x \end{bmatrix} = \int \frac{\frac{\mathrm{d}u}{2}}{\sqrt{u}} = \frac{1}{2}\int u^{-\frac{1}{2}}\mathrm{d}u =$

$\displaystyle\frac{1}{2}\frac{1}{\frac{1}{2}}u^{\frac{1}{2}} + C = \sqrt{u} + C = \sqrt{x^2 - a^2} + C.$ □

3 Integrace metodou per partes

$$(u \cdot v)' = u'v + uv' \quad \Longrightarrow \quad uv' = (u \cdot v)' - u'v \quad \Longrightarrow$$

$$\int uv'\,\mathrm{d}x = \int (u \cdot v)'\,\mathrm{d}x - \int u'v\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x.$$

Úloha 3.1. $I = \displaystyle\int \sin^2 x\,\mathrm{d}x$.

Řešení. $I = \displaystyle\int \sin^2 x\,\mathrm{d}x = \begin{bmatrix} u = \sin x & v' = \sin x \\ u' = \cos x & v = -\cos x \end{bmatrix} = -\sin x \cos x + \int \cos^2 x\,\mathrm{d}x =$

$-\sin x\cos x + \displaystyle\int (1 - \sin^2 x)\,\mathrm{d}x = -\sin x \cos x + \int \mathrm{d}x - \int \sin^2 x\,\mathrm{d}x = -\sin x \cos x +$

$x - \displaystyle\int \sin^2 x\,\mathrm{d}x.$

Máme tedy rovnost:

$$\int \sin^2 x \,\mathrm{d}x = -\sin x \cos x + x - \int \sin^2 x \,\mathrm{d}x.$$

Odtud:

$$\int \sin^2 x \,\mathrm{d}x = \frac{x - \sin x \cos x}{2} + C.$$

□

Úloha 3.2. $I = \displaystyle\int x^2 \cos 2x \,\mathrm{d}x.$

Řešení. $\displaystyle\int x^2 \cos 2x \,\mathrm{d}x = \begin{bmatrix} u = x^2 & v' = \cos 2x \\ u' = 2x & v = \frac{1}{2}\sin 2x \end{bmatrix} =$

$\displaystyle x^2 \frac{1}{2} \sin 2x - \int 2x \frac{1}{2} \sin 2x \,\mathrm{d}x = \frac{x^2}{2} \sin 2x - \int x \sin 2x \,\mathrm{d}x =$

$\displaystyle \begin{bmatrix} u = x & v' = \sin 2x \\ u' = 1 & v = -\frac{1}{2}\cos 2x \end{bmatrix} = \frac{x^2}{2} \sin 2x + x \frac{1}{2} \cos 2x - \frac{1}{2} \int \cos 2x \,\mathrm{d}x =$

$\displaystyle \frac{x^2}{2} \sin 2x + \frac{x}{2} \cos 2x - \frac{1}{2}\frac{1}{2} \sin 2x + C = \frac{x^2}{2} \sin 2x + \frac{x}{2} \cos 2x - \frac{1}{4} \sin 2x + C.$ □

Úloha 3.3. $I = \displaystyle\int \mathrm{tg}^2\, x \,\mathrm{d}x.$

Řešení. $I = \displaystyle\int \mathrm{tg}^2\, x \,\mathrm{d}x = \int \frac{\sin^2 x}{\cos^2 x} \,\mathrm{d}x =$

$\displaystyle \begin{bmatrix} u = \sin^2 x & v' = \frac{1}{\cos^2 x} \\ u' = 2 \sin x \cos x & v = \mathrm{tg}\, x = \frac{\sin x}{\cos x} \end{bmatrix} = \sin^2 x \frac{\sin x}{\cos x} - \int 2 \sin x \cos x \frac{\sin x}{\cos x} \,\mathrm{d}x =$

$\displaystyle \frac{\sin^3 x}{\cos x} - 2 \int \sin^2 x \,\mathrm{d}x = \begin{bmatrix} \displaystyle\int \sin^2 x \,\mathrm{d}x = \frac{x - \sin x \cos x}{2} + C \\ \text{(Úloha 3.1)} \end{bmatrix} =$

$\displaystyle \frac{\sin^3 x}{\cos x} - 2 \frac{x - \sin x \cos x}{2} + C = \frac{\sin^3 x}{\cos x} - x + \sin x \cos x + C.$ □

3.1 Výpočet I_c a I_s

Úloha 3.4. *Vypočtěte*

$$I_c = \int \mathrm{e}^{ax} \cos bx \ \mathrm{d}x, \quad I_s = \int \mathrm{e}^{ax} \sin bx \ \mathrm{d}x$$

(budeme počítat primitivní funkce pro $C = 0$).

Řešení. V integrálu I_c se použije dvěma způsoby metoda per partes: a) pro $u = \cos bx$ a $v' = \mathrm{e}^{ax}$, b) pro $u = \mathrm{e}^{ax}$, $v' = \cos bx$:

a) $I_c = \displaystyle\int \mathrm{e}^{ax} \cos bx \,\mathrm{d}x = \left[\begin{array}{ll} u = \cos bx & v' = \mathrm{e}^{ax} \\ u' = -b \sin bx & v = \frac{1}{a}\,\mathrm{e}^{ax} \end{array} \right] = \frac{1}{a}\,\mathrm{e}^{ax} \cos bx + \frac{b}{a} \int \mathrm{e}^{ax} \sin bx \,\mathrm{d}x = \frac{1}{a}\,\mathrm{e}^{ax} \cos bx + \frac{b}{a} I_s.$

b) $I_c = \displaystyle\int \mathrm{e}^{ax} \cos bx \,\mathrm{d}x = \left[\begin{array}{ll} u = \mathrm{e}^{ax} & v' = \cos bx \\ u' = a\,\mathrm{e}^{ax} & v = \frac{1}{b} \sin bx \end{array} \right] = \frac{1}{b}\,\mathrm{e}^{ax} \sin bx - \frac{a}{b} \int \mathrm{e}^{ax} \sin bx \,\mathrm{d}x = \frac{1}{b}\,\mathrm{e}^{ax} \sin bx - \frac{a}{b} I_s.$

Tím dostaneme soustavu

$$I_c = \frac{1}{a}\,\mathrm{e}^{ax} \cos bx + \frac{b}{a} I_s, \quad I_c = \frac{1}{b}\,\mathrm{e}^{ax} \sin bx - \frac{a}{b} I_s,$$

tedy

$$\begin{aligned} \frac{1}{a}\,\mathrm{e}^{ax} \cos bx + \frac{b}{a} I_s &= \frac{1}{b}\,\mathrm{e}^{ax} \sin bx - \frac{a}{b} I_s, \\ \frac{b}{a} I_s + \frac{a}{b} I_s &= \frac{1}{b}\,\mathrm{e}^{ax} \sin bx - \frac{1}{a}\,\mathrm{e}^{ax} \cos bx \quad | \cdot ab, \\ (b^2 + a^2) I_s &= a\,\mathrm{e}^{ax} \sin bx - b\,\mathrm{e}^{ax} \cos bx, \end{aligned}$$

odtud

$$\begin{aligned} I_s &= \frac{a \sin bx - b \cos bx}{a^2 + b^2}\,\mathrm{e}^{ax}. \\ I_c &= \frac{1}{a}\,\mathrm{e}^{ax} \cos bx + \frac{b}{a} I_s = \frac{1}{a}\,\mathrm{e}^{ax} \cos bx + \frac{b}{a} \frac{a \sin bx - b \cos bx}{a^2 + b^2}\,\mathrm{e}^{ax} \\ &= \frac{(a^2 + b^2) \cos bx + ab \sin bx - b^2 \cos bx}{a(a^2 + b^2)}\,\mathrm{e}^{ax} = \frac{b \sin bx + a \cos bx}{a^2 + b^2}\,\mathrm{e}^{ax} \end{aligned}$$

□

3.2 Rekurentní vzorec pro integrál I_n

Úloha 3.5. *Nalezněte rekurentní vzorec pro integrál* $I_n = \displaystyle\int \frac{\mathrm{d}x}{(a^2 + x^2)^n}$, $n \geq 2$.

Řešení. V integrálu I_m, kde $m \geq 1$, položíme $u = \dfrac{1}{(a^2 + x^2)^n}$, $v' = 1$ a dostaneme $I_m = \dfrac{x}{(a^2 + x^2)^m} + 2mI_m - 2ma^2 I_{m+1}$, odkud vyjádříme I_{m+1}. Položíme-li pak $m = n - 1$, dostaneme

$$I_n = \frac{1}{2(n-1)a^2} \, \frac{x}{(a^2 + x^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1}.$$

□

Úloha 3.6. *Vypočtěte* $I = \displaystyle\int \frac{\mathrm{d}x}{\left(2^2 + x^2\right)^3}$.

Řešení. Využijeme rekurentní vzorec pro I_3:

$$
\begin{aligned}
I &= I_3 = \frac{1}{2(3-1)2^2}\,\frac{x}{\left(2^2+x^2\right)^{3-1}} + \frac{2\cdot 3-3}{2(3-1)2^2} I_{3-1} \\
&= \frac{1}{16}\,\frac{x}{\left(2^2+x^2\right)^2} + \frac{3}{16}\int \frac{\mathrm{d}x}{\left(2^2+x^2\right)^2} \\
&= \frac{1}{16}\,\frac{x}{\left(2^2+x^2\right)^2} + \frac{3}{16}\left(\frac{1}{2(2-1)2^2}\,\frac{x}{\left(2^2+x^2\right)^{2-1}} + \frac{2\cdot n-3}{2(2-1)2^2} I_{2-1}\right) \\
&= \frac{1}{16}\,\frac{x}{\left(2^2+x^2\right)^2} + \frac{3}{16}\left(\frac{1}{8}\,\frac{x}{2^2+x^2} + \frac{1}{8}\int \frac{\mathrm{d}x}{2^2+x^2}\right) \\
&= \frac{1}{16}\,\frac{x}{\left(2^2+x^2\right)^2} + \frac{3}{16}\left(\frac{1}{8}\,\frac{x}{2^2+x^2} + \frac{1}{8}\left(\frac{1}{2^2}\int \frac{\mathrm{d}x}{1+\left(\frac{x}{2}\right)^2}\right)\right) \\
&= \frac{1}{16}\,\frac{x}{\left(2^2+x^2\right)^2} + \frac{3}{16}\left(\frac{1}{8}\,\frac{x}{2^2+x^2} + \frac{1}{8}\left(\frac{1}{2^2}\,\frac{1}{\frac{1}{2}}\operatorname{arctg}\frac{x}{2}\right)\right) + C \\
&= \frac{1}{16}\left(\frac{x}{\left(2^2+x^2\right)^2} + \frac{3}{8}\,\frac{x}{2^2+x^2} + \frac{3}{16}\operatorname{arctg}\frac{x}{2}\right) + C.
\end{aligned}
$$

□