Lineární DR 1. řádu – příklady

1a) $y' - y = e^{2x}$ ⊛

HÚ: $y' - y = 0$, $p(x) \equiv -1$, $x \in \mathbb{R}$, $y \in \mathbb{R}$

ØŘHÚ: $y = C \cdot e^{-\int p(x)dx} = C \cdot e^{\int dx} = C \cdot e^{x}$, $C \in \mathbb{R}$

NHÚ: $y = K(x) \cdot e^{x}$, $y' = K'(x) \cdot e^{x} + K(x) e^{x}$

Dosadíme do ⊛

$K'(x) e^{x} + \underline{K(x) e^{x}} - \underline{K(x) e^{x}} = e^{2x}$, $K'(x) e^{x} = e^{2x} \quad | : e^{x}$

$K'(x) = e^{x}$, tedy $K(x) = e^{x} + C$

ØŘNHÚ: $\boxed{y = K(x) \cdot e^{x} = (e^{x} + C) \cdot e^{x} = \underline{\underline{e^{2x} + C e^{x}}}, \quad x \in \mathbb{R}, C \in \mathbb{R}}$

1b) $y' - \frac{x+1}{x} y = x^2$, $x \neq 0$, $y \in \mathbb{R}$

HÚ: $y' - \frac{x+1}{x} y = 0$, $p(x) = -\frac{x+1}{x}$, $-\int p(x)dx = \int \frac{x+1}{x} dx = \int (1 + \frac{1}{x}) dx$
$= x + \ln|x| + C$

ØŘHÚ: $y = C \cdot e^{-\int p(x)dx} = C \cdot e^{x + \ln|x|} = C \cdot |x| \cdot e^{x} \overset{⊞}{=} C \cdot x \cdot e^{x}$, $C \in \mathbb{R}$, $x \neq 0$

NHÚ: $y = K(x) \cdot x \cdot e^{x}$, $y' = K'(x) \cdot x \cdot e^{x} + K(x) e^{x} + K(x) x e^{x}$

dosadíme:

$K'(x) \cdot x \cdot e^{x} + K(x) e^{x} + K(x) x e^{x} - \underbrace{\frac{x+1}{x}}_{1 + \frac{1}{x}} \cdot K(x) \cdot x \cdot e^{x} = x^2$

$K'(x) \cdot x \cdot e^{x} + \underline{K(x) e^{x}} + \underline{K(x) x e^{x}} - \underline{1 \cdot K(x) \cdot x \cdot e^{x}} - \underline{\frac{1}{\cancel{x}} K(x) \cancel{x} e^{x}} = x^2$

$K'(x) \cdot x \cdot e^{x} = x^2$, $K'(x) = \frac{x}{e^{x}} = x \cdot e^{-x}$

$K(x) = \int x \cdot e^{-x} dx = \begin{bmatrix} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{bmatrix} = x \cdot (-e^{-x}) - \int -e^{-x} dx = -x e^{-x} - e^{-x} + C$

ØŘNHÚ: $y = K(x) \cdot x \cdot e^{x} = (-x e^{-x} - e^{-x} + C) \cdot x \cdot e^{x} = -x^2 - x + C x e^{x}$

$\boxed{y = -x^2 - x + C x e^{x}, \quad C \in \mathbb{R}, \ x \neq 0}$

⊞ x je buď < 0 ($\in (-\infty, 0)$) nebo > 0 ($\in (0, +\infty)$), a tak $C \cdot |x|$, $C \in \mathbb{R}$ reprezentuje stejné funkce jako $C x$.

1c) $(x^2+1)y' - 2xy = x^2$ | dělíme výrazem (x^2+1), který je >0

$$y' - \frac{2x}{x^2+1}y = \frac{x^2}{x^2+1}, \quad x\in\mathbb{R},\ y\in\mathbb{R}$$

HÚ: $y' - \frac{2x}{x^2+1}y = 0$, $p(x) = -\frac{2x}{x^2+1}$, $-\int p(x)dx = \int \frac{2x}{x^2+1}dx = \ln|x^2+1| + C$

$(x^2+1)' = 2x$ $\quad = \ln(x^2+1)+C$

OŘHÚ: $y = C\cdot e^{-\int p(x)dx} = C\cdot e^{\ln(x^2+1)} = C\cdot \cancel{\ln}(x^2+1)$, $C\in\mathbb{R}, x\in\mathbb{R}$

NHÚ: $y = K(x)\cdot\cancel{\ln}(x^2+1)$, $y' = K'(x)\cdot\cancel{\ln}(x^2+1) + K(x)\cdot \cancel{\frac{2x}{x^2+1}}$

dosadíme:

$$K'(x)\cdot\cancel{\ln}(x^2+1) + \underline{K(x)\cdot\frac{2x}{\cancel{x^2+1}}} - \underline{\frac{2x}{\cancel{x^2+1}}K(x)\cancel{(x^2+1)}} = \frac{x^2}{x^2+1}$$

$K'(x)(x^2+1) = \frac{x^2}{x^2+1}$, $K'(x) = \frac{x^2}{(x^2+1)^2}$

$$K(x) = \int \frac{x^2}{(x^2+1)^2}dx = \begin{bmatrix} u = x & v' = \frac{x}{(x^2+1)^2} \\ u' = 1 & v = -\frac{1}{2}\frac{1}{1+x^2}\end{bmatrix} =$$

$$\left[\int\frac{x}{(x^2+1)^2}dx = \begin{bmatrix} x^2+1 = u \\ 2x\,dx = du\end{bmatrix} = \frac{1}{2}\int\frac{2x\,dx}{(x^2+1)^2} = \frac{1}{2}\int\frac{du}{u^2} = \frac{1}{2}\int u^{-2}du = \frac{1}{2}\frac{u^{-1}}{-1}+C = -\frac{1}{2}\frac{1}{u}+C = -\frac{1}{2}\frac{1}{1+x^2}+C\right]$$

$$= -\frac{1}{2}x\cdot\frac{1}{1+x^2} + \frac{1}{2}\int\frac{dx}{1+x^2} = -\frac{1}{2}\frac{x}{1+x^2} + \frac{1}{2}\operatorname{arctg} x + C$$

OŘNHÚ: $y = K(x)\cdot(x^2+1) = \left(-\frac{1}{2}\frac{x}{1+x^2} + \frac{1}{2}\operatorname{arctg} x + C\right)\cdot(x^2+1)$

$$\boxed{y = -\frac{x}{2} + \frac{1}{2}(x^2+1)\operatorname{arctg} x + C(x^2+1),\quad C\in\mathbb{R},\ x\in\mathbb{R}}$$

Takový příklad na písemce nebude.

2a) $x \cdot y' - y = \sqrt{x}$, $y(4) = 12$

$\downarrow$
$x \geq 0$

Pro $x=0$ dostaneme $0 \cdot y'(0) - y(0) = \sqrt{0}$, $-y(0) = 0$, $y(0) = 0$,
takže všechna řešení mají pro $x=0$ hodnotu 0.
Pokud rovnici dělíme x, abychom dostali standardní tvar, musíme vyloučit $x=0$.
Tedy pro $x>0$ máme

$$y' - \frac{y}{x} = \frac{\sqrt{x}}{x} \quad , p(x) = \frac{-1}{x} \ , \ q(x) = \frac{\sqrt{x}}{x}$$

HÚ: $y' - \frac{y}{x} = 0$, OŘHÚ

$$y = C \cdot e^{-\int \frac{-dx}{x}} = C \cdot e^{\ln|x|} = C \cdot |x| = Cx, C \in \mathbb{R}$$
$$(x>0)$$

$$y = C \cdot x, \ x>0, C \in \mathbb{R}$$

NHÚ: OŘ $y = K(x) \cdot x$, $y' = K'(x) \cdot x + K(x)$

dosadíme: $$\underbrace{K'(x) \cdot x + K(x)} - \underbrace{\frac{K(x) \cdot \cancel{x}}{\cancel{x}}} = \frac{\sqrt{x}}{x}$$

$$K'(x) \cdot x = \frac{\sqrt{x}}{x} \ , \quad K'(x) = \frac{\sqrt{x}}{x^2} = x^{\frac{1}{2}-2} = x^{-\frac{3}{2}}$$

$$K(x) = \int x^{-\frac{3}{2}} dx = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + C = -2 \cdot \frac{1}{\sqrt{x}} + C$$

OŘNHÚ: $y = K(x) \cdot x = \left(-\frac{2}{\sqrt{x}} + C\right) \cdot x$

$$y = -2\sqrt{x} + Cx, \quad x>0, \quad C \in \mathbb{R}$$

$\lim\limits_{x \to 0^+} (-2\sqrt{x} + Cx) = 0$, tudíž všechny tyto funkce splňují podmínku z úvodu pro $x=0$, tedy $y(0)=0$, a tak máme řešení

$$\left[y = -2\sqrt{x} + Cx \ , \ x \geq 0, \ C \in \mathbb{R} \right]$$

2a - pokračování - počáteční podmínka)

$$y(4) = 12$$

$$-2\sqrt{4} + C\cdot 4 = 12$$

$$-4 + C\cdot 4 = 12$$

$$C\cdot 4 = 16$$

$$\underline{C = 4} \Rightarrow \left[y = -2\sqrt{x} + 4x,\ x \geq 0 \right]$$

2b $y' + y \operatorname{cotg} x = 1,\quad y\left(\frac{\pi}{2}\right) = 0,$ $\quad \sin x \neq 0$, $x \neq k\pi, k \in \mathbb{Z}$

HÚ: $y' + y \cdot \operatorname{cotg} x = 0$, OŘHÚ $y = C \cdot e^{-\int \operatorname{cotg} x\, dx} =$

$$y = C\cdot e^{-\int \frac{\cos x}{\sin x} dx} = C \cdot e^{-\ln|\sin x|} = C\cdot e^{\ln\frac{1}{|\sin x|}} = C\cdot \frac{1}{|\sin x|},\ x \neq k\pi,\ C \in \mathbb{R}$$

$= C \cdot \frac{1}{\sin x}$, (neboť $\sin x$ na daných intervalech nemění znaménko)

NHÚ: $y = \frac{K(x)}{\sin x}$, $y' = \frac{K'(x)\sin x - K(x)\cos x}{\sin^2 x}$

dosadíme:

$$\frac{K'(x)}{\sin x} \underbrace{- K(x)\cdot\frac{\cos x}{\sin^2 x} + \frac{K(x)}{\sin x}\cdot \overset{\operatorname{cotg} x}{\frac{\cos x}{\sin x}}} = 1$$

$K'(x) = \sin x$, $\quad K(x) = -\cos x + C$

OŘNHÚ: $y = \frac{K(x)}{\sin x} = \frac{-\cos x + C}{\sin x}$, $\left[y = -\operatorname{cotg} x + \frac{C}{\sin x},\ C \in \mathbb{R},\ x \neq k\pi, k \in \mathbb{Z} \right]$

Počáteční úloha: $y\left(\frac{\pi}{2}\right) = 0$, $-\operatorname{cotg}\frac{\pi}{2} + \frac{C}{\sin\frac{\pi}{2}} = 0$, $0 + \frac{C}{1} = 0$

$C = 0$

$$\left[y = -\operatorname{cotg} x,\ x \in (0, \pi) \right] \left(\text{neboť } \tfrac{\pi}{2} \in (0,\pi)\right)$$

(2c) $x^2 y' + xy = \ln x, \quad y(1) = \frac{1}{2},$

$\Updownarrow$ $x > 0$ – můžeme tedy „bez obav" dělit rovnici x^2

$$y' + \frac{y}{x} = \frac{\ln x}{x^2}, \quad y(1) = \frac{1}{2}$$

HÚ: $y' + \frac{y}{x} = 0, \quad p(x) = \frac{1}{x}, \quad -\int p(x)dx = \int -\frac{1}{x}dx = -\ln|x| + C$

OŘHÚ: $y = C \cdot e^{-\ln|x|} = C \cdot e^{\ln \frac{1}{|x|}} = C \cdot \frac{1}{\underset{x>0}{|x|}} = \frac{C}{x}, \quad x > 0, \; C \in \mathbb{R}$

$$\left[y = C \cdot \frac{1}{x}, \; C \in \mathbb{R}, \; x > 0 \right]$$

NHÚ: $y = \frac{K(x)}{x}, \quad y' = \frac{K'(x) \cdot x - K(x)}{x^2} = \frac{K'(x)}{x} - \frac{K(x)}{x^2}$

dosadíme:

$$\frac{K'(x)}{x} - \frac{K(x)}{x^2} + \frac{\frac{K(x)}{x}}{x} = \frac{\ln x}{x^2}$$

$K'(x) = \ln x, \quad K(x) = \int \frac{\ln x}{x} dx = \left[\begin{matrix} u = \ln x, & \cancel{u'=1} \; du = \frac{1}{x}dx \\ \cancel{v' = 1} & \cancel{v = x} \end{matrix} \right] =$

$= \int u \, du = \frac{u^2}{2} + C = \frac{\ln^2 x}{2} + C = \cancel{x \ln x - \int 1 dx = x \ln x - x + C}$

OŘNHÚ: $y = \frac{K(x)}{x} = \cancel{\frac{x \ln x - x + C}{x} = \ln x - 1 + \frac{C}{x}} = \frac{\frac{\ln^2 x}{2}}{x} + \frac{C}{x}$

$$\left[y = \frac{\ln^2 x}{2x} \cancel{+1} + \frac{C}{x}, \quad x > 0, \; C \in \mathbb{R} \right]$$

Počáteční úloha: $y(1) = \frac{1}{2}$

$\frac{\ln^2 1}{2 \cdot 1} + \frac{C}{1} = \frac{1}{2}, \quad C = \frac{1}{2}$ $\qquad$ $\cancel{\ln 1 - 1 + \frac{C}{1} = \frac{1}{2}, \quad 0 - 1 + C = \frac{1}{2}}$

$\cancel{C = \frac{1}{2} + 1}$

$y = \frac{\ln^2 x}{2x} + \frac{1}{2x}$ $\qquad$ $\cancel{C = \frac{3}{2}}$

$\left[y = \frac{1 + \ln^2 x}{2x}, \; x > 0 \right]$ $\qquad$ $\cancel{\left[y = \ln x - 1 + \frac{3}{2x}, \; x > 0 \right]}$